CENTRE
DE RECHERCHE
SUR LES TRANSPORTS
(CRT)
CENTRE
FOR RESEARCH
ON TRANSPORTATION

UNIVERSITÉ DE MONTRÉAL
ÉCOLE DES HAUTES ÉTUDES COMMERCIALES
ÉCOLE POLYTECHNIQUE

C.P. 6128, SUCCURSALE A
MONTRÉAL (QUÉBEC)
CANADA
H3C 3J7

TÉLÉPHONE : (514) 343-7575
TÉLEX : 05-24146 BIBPOLYTEC
TÉLÉCOPIEUR (FAX) : (514) 343-7121

CRT-631

VEHICLE SCHEDULING PROBLEM: LAGRANGEAN DECOMPOSITION AND DUALITY GAP

par

Philippe Michelon*
Jean-Pierre Crouzeix*
Jacques A. Ferland**

* Département de mathématiques appliquées, Université de Clermont II, France.

** Département d'informatique et de recherche opérationnelle (DIRO), Université de Montréal.

This research was supported by a NSERC grant (A8312) and the Scientific Exchange Program between France and Québec (grant 20-02-13-88).

Université de Montréal

Centre de recherche sur les transports - Publication # 631
Laboratoire d'optimisation du DIRO - Publication # LO 89-1
Département d'informatique et de recherche opérationnelle - Publication # 676

Février 1989

Dépôt légal - 2e trimestre 1989

Abstract: Lagrangean decomposition and Lagrangean relaxation are used to deal with the vehicle scheduling problem with multiple vehicle types and time windows. Relying on general results about Lagrangean decomposition, it is shown that for many vehicle scheduling problems, lower bounds generated with Lagrangean decomposition and Lagrangean relaxation are identical. Furthermore, applying general results about duality gap to our problem, it is shown that it can be reduced to zero by using augmented versions of Lagrangean decomposition and relaxation.

Key words: Vehicle scheduling, time windows, Lagrangean decomposition, Lagrangean relaxation, duality gap.

Résumé: Le problème d'horaires de véhicules avec fenêtres de temps où plusieurs types de véhicules sont utilisés est traité avec la décomposition et la relaxation lagrangienne. Se référant à des résultats généraux sur la décomposition lagrangienne, on démontre que pour plusieurs problèmes d'horaires de véhicules, les bornes inférieures générées avec la décomposition et la relaxation lagrangienne sont identiques. De plus, des résultats généraux sur la marge duale nous permettent de démontrer que, pour notre problème, celle-ci peut être réduite à zéro en utilisant la relaxation et la décomposition lagrangienne augmentée.

Mots clés: Horaires de véhicules, fenêtres de temps, décomposition lagrangienne, relaxation lagrangienne, marge duale.

1. INTRODUCTION

The vehicle scheduling problem can be specified in terms of a set of tasks T to be executed with a fleet of vehicles. Each task i ($1 \leq i \leq n$) has time and space (location) characteristics specified as follows:

- a start location SL_i
- an end location EL_i
- a start time ST_i .

In most realistic contexts, the starting times ST_i are not fixed but rather belong to some given time intervals [a_i , b_i] referred to as "time windows". Hence, for each task i, a_i and b_i are the earliest and the latest starting times, respectively. It is also assumed that task i can be executed only by a vehicle having a capacity exceeding q_i .

The fleet of vehicles available can be characterized by its size (the number of vehicles) and by the types of vehicles. We assume that several types of vehicles are available. Each type k ($1 \leq k \leq K$) of vehicles is characterized by its capacity h_k and its speed v_k .

To execute task i , a vehicle of type k must be at location SL_i at time ST_i. Assume that a vehicle of type k requires D_i^k units of time to execute task i. Then at time $ST_i + D_i^k$, it is located at EL_i and available to execute other tasks. The problem is to partition the set of tasks into sequences. Since all tasks of a sequence are to be completed by the same vehicle, it follows that no two of these can be executed simultaneously and the vehicle type selected for the sequence must be acceptable for all of its tasks. Furthermore, task j can follow task i in a sequence executed by a vehicle of type k if, taking into account the time t_{ij}^k required for the vehicle to go from EL_i to SL_j, it is available at time ST_j at SL_j ;

i.e.,

$$ST_i + D_i^k + t_{ij}^k \leq ST_j$$

where $t_{ij}^k + D_i^k$ depends on the vehicle speed v_k.

The objective function to be minimized can be specified in terms of operating costs, or in terms of investment costs, or in terms of both. For instance, the objective might be to reduce the fleet size to a minimum.

The vehicle scheduling problem is presented and analyzed in Bodin et al. [1] where several procedures are summarized. More recently the vehicle scheduling problem with time windows has been analyzed by several authors [2, 3, 4, 5, 6, 11, 13, 14, 15]. In [7] Ferland summarizes the main approaches to handle this problem when a single type of vehicles is used. Several of these procedures are extended to handle the problem with multiple vehicle types, in [8].

The problem is formulated in Section 2. Then the notion of Lagrangean decomposition [12] is introduced in Section 3. In general, a Lagrangean decomposition generates a lower bound on optimal value which is tighter than the one obtained with a Lagrangean relaxation ([9, 10]). But applying these results to the vehicle scheduling problem, it is shown in Section 4, that in many cases the bounds generated with Lagrangean decomposition and relaxation are identical. Finally, in Section 5, duality gap is analyzed in a general context. It is shown that, for some problems, the duality gap can be reduced to zero if the augmented version of a Lagrangean decomposition or relaxation is used with a penalty coefficient large enough. The vehicle scheduling problem falls in this class, and hence, this is a theoretical justification of what Desrosiers et al. [6] observed in their numerical results when using the augmented Lagrangean relaxation to deal with the problem with a unique type of vehicle (i.e., K = 1).

2. PROBLEM FORMULATION

A mathematical formulation is derived for this problem using the quasi-precedence graphs (N_k , A_k) of the tasks associated with vehicle types k and characterized as follows:

i) With each task i such that $q_i \le h_k$, associate a node $i \in N_k$. Two nodes, s and t, are associated with the depot to indicate where the vehicles start and finish.

ii) An arc (i, j), exits (i.e. $(i, j) \in A_k$, or task j can be executed after task i by a vehicle of type k) if

$$a_i + D_i^k + t_{ij}^k \le b_j .$$

Furthermore, for each task $i \in N_k$, introduce arcs (s, i) and (i, t) in A_k allowing i to be the first or the last task of a sequence, respectively.

In terms of these oriented graphs the problem is to partition the nodes into a set of paths (or sequences), each of these starting at s and ending at t in one of the graphs (N_k , A_k). The time window restrictions are to be satisfied for all tasks, and the objective is to minimize the total cost. The formulation is summarized as follows:

(P) Min $\sum_{k=1}^{K} \sum_{(i,j)\in A_k} c_{ij}^k x_{ij}^k$

Subject to

$$\sum_{k=1}^{K} \sum_{i\in N_k} x_{ij}^k = 1 \qquad j \in N_k - \{s,t\}, \qquad (2.1)$$

$$\sum_{i\in P_j^k} x_{ij}^k = \sum_{i\in S_j^k} x_{ji}^k \qquad j \in N_k - \{s,t\}, \quad 1 \le k \le K \qquad (2.2)$$

$$x_{ij}^k > 0 \Rightarrow ST_i + D_i^k + t_{ij}^k \le ST_j \qquad (i, j) \in \bar{A}_k , \quad 1 \le k \le K \qquad (2.3)$$

$$a_i \le ST_i \le b_i \qquad 1 \le i \le n \qquad (2.4)$$

cycle-elimination constraints (2.5)

$$x_{ij}^k = 0 \text{ or } 1 \qquad (i, j) \in A_k , \quad 1 \le k \le K \qquad (2.6)$$

where

$\bar{A}_k = \{(i, j) \in A_k: i \neq s, t; j \neq s, t\}$, $P_j^k = \{i \in N_k : (i, j) \in A_k\}$ and $S_j^k = \{i \in N_k : (j, i) \in A_k\}$.

Furthermore the decision variables are defined as follows:

$$x_{ij}^k = \begin{cases} 1 & \text{if } \textit{a vehicle of type } k \textit{ traverses arc } (i, j) \in A_k \\ 0 & \textit{otherwise} \end{cases}$$

The linear costs c_{ij}^k are specified according to the objective of the problem. For instance, if the objective is to minimize the investment cost, then:

$$c_{ij}^k = \begin{cases} u^k & \text{if } i = s \\ 0 & \text{if } i \neq s \end{cases}$$

where u^k is equal to the investment cost for vehicle type k. If the objective is to minimize investment and running time, then:

$$c_{ij}^k = \begin{cases} D_i^k + t_{ij}^k & \text{if } i \text{ and } j \textit{ are different from } s \textit{ and } t \\ u^k & \text{if } i = s \\ D_i^k & \text{if } j = s \end{cases}$$

where the magnitude of u^k relative to t_{ij}^k and D_i^k measures the importance of investment versus running time.

Constraints (2.1) and (2.2) ensure that each task is completed by exactly one vehicle. Constraints (2.3) and (2.4) are the time window constraints. Finally, additional linear constraints are included in (2.5) to eliminate cycle where a path does not start and end at the depot.

3. LAGRANGEAN DECOMPOSITION

The Lagrangean decomposition approach is presented in [12]. Like Lagrangean relaxation, this approach is very appropriate to deal with problems where the set of constraints can be partitioned into two subsets. To introduce the approach, consider the mathematical programming problem:

(Ω)

$$\underset{x \in X_1 \cap X_2 \cap X}{Min} \; f(x)$$

where X , X_1 and X_2 are subsets of R^n and f: $R^n \rightarrow R^1$. This problem is equivalent to:

(Ω')

$$\begin{aligned} \text{Min} \quad & f(x) \\ \text{Subject to} \quad & x \in X_1 \cap X \\ & y \in X_2 \cap \text{Y} \\ & y - x = 0 \end{aligned}$$

where Y includes X (*i.e.*, $X \subset$ Y).

The Lagrangean decomposition of (Ω) corresponds to the Lagrangean relaxation of (Ω') relative to the constraints $y - x = 0$ and the associated vector of multipliers u :

(Ω' R_u)

$$\begin{aligned} v_D(u) = Min \quad & f(x) + u^T (y - x) \\ \text{Subject to} \quad & x \in X_1 \cap X \\ & y \in X_2 \cap \text{Y} \end{aligned}$$

This problem is separable and can be formulated as:

$(\Omega'\ R_u)$

$$v_D(u) = \underset{x \in X_1 \cap X}{Min} \{f(x) - u^T x\} + \underset{y \in X_2 \cap Y}{Min} \{u^T y\}.$$

Clearly, (taking $y = x$) for all u:

$$v_D(u) \leq v\ (\Omega') = v\ (\Omega) \tag{3.1}$$

where $v(\Omega')$ and $v(\Omega)$ denote the optimal value of (Ω') and (Ω), respectively.

Furthermore, suppose that the constraints $x \in X$ imply that some components of x have to be integer (i.e., (Ω) is a mixed integer problem). Denote $(\overline{\Omega})$ and $(\overline{\Omega}')$ the problem where the integrality constraints are relaxed. Then (see [10]):

$$v(\overline{\Omega}) \leq \underset{u}{Max}\ v_D(u) \leq v(\Omega)$$

(where $v(\overline{\Omega})$ is the optimal value of $(\overline{\Omega})$) .

Now, assume that X_2 is a polyhedral convex set of the form $X_2 = \{x \in R^n: Ax \leq b\}$. The Lagrangean relaxation of (Ω) relative to the constraints $Ax \leq b$ and the associated vector of multipliers λ is specified as follows:

$(\Omega\ R_\lambda)$

$$v_R(\lambda) = \underset{x \in X_1 \cap X}{Min} f(x) + \lambda^T (Ax - b).$$

Then,

$$v_R(\lambda) \leq v(\Omega). \tag{3.2}$$

Since Lagrangean decomposition and Lagrangean relaxation generate lower bounds on $v(\Omega)$, (see (3.1) and (3.2)), it is legitimate to see if one is tighter than the other.

The following result generalizes Theorem 3.1 of Guignard and Kim [12].

THEOREM 3.1

a- Assume that $X_2 = \{x \in R^n: Ax \leq b\}$. Then

$$v_R(\lambda) \leq v_D(-A^T\lambda) \text{ for all } \lambda \geq 0.$$

It follows that:

$$\max_{\lambda \geq 0} \{v_R(\lambda)\} \leq \max_{u} \{v_D(u)\}$$

b- If in addition, X_2 is compact and $CO(X_2 \cap Y) = CO(X_2)$, then

$$\max_{\lambda \geq 0} \{v_R(\lambda)\} = \max_{u} \{v_D(u)\}$$

PROOF

a- $$v_D(-A^T\lambda) = \min_{x \in X_1 \cap X} \{f(x) + \lambda^T Ax\} - \max_{y \in X_2 \cap Y} \lambda^T Ay \, .$$

But $\max_{y \in X_2 \cap Y} \{\lambda^T Ay\} \leq \max_{y \in X_2} \{\lambda^T Ay\}$, and furthermore $Ay \leq b$ and $\lambda \geq 0$ imply $\lambda^T Ay \leq b^T\lambda$. Hence $v_D(-A^T\lambda) \geq v_R(\lambda)$ for all $\lambda \geq 0$.

b- Since $CO(X_2)$ is compact (X_2 compact $\Rightarrow$ $CO(X_2)$ compact) then for all u , one has:

$$Min\{u^T y : Ay \leq b\} = \max\{-\lambda^T b : -A^T\lambda = u,\ \lambda \geq 0\}.$$

Let $\lambda(u)$ be an optimal solution of the maximization problem (such a λ exists). Then

$$\min_{y \in X_2 \cap Y} \{u^T y\} = \min_{Y \in CO(X_2 \cap Y)} \{u^T y\} = \min_{y \in CO(X_2)} \{u^T y\} = -\lambda(u)^T b.$$

Replacing u by $-A^T\lambda(u)$, we obtain

$$v_D(u) = \min_{x \in X_1 \cap X} \{f(x) + \lambda(u)^T (Ax - b)\} = v_R(\lambda(u)).$$

Hence $\max_u v_D(u) = \max_{\lambda \geq 0} v_R(\lambda)$.

□

Hence, Lagrangean decomposition generates a lower bound at least as good as Lagrangean relaxation. In the next section, we apply Lagrangean decomposition to the vehicle scheduling problem.

4. LAGRANGEAN DECOMPOSITION APPLIED TO VEHICLE SCHEDULING PROBLEM

Referring to the vehicle scheduling problem with time windows (P) specified in Section 2, several different decompositions can be derived according to the ways X_1 and X_2 are specified.

The first decomposition (D_1) is obtained with:

$$X_1 = \{x : (2.1) \text{ are verified and } x \geq 0\}$$
$$X_2 = \{x : (2.2) \text{ to } (2.5) \text{ are verified}\}$$
$$X = \{x : x_{ij}^k = 0 \text{ or } 1\}$$
$$Y = \{y : y_{ij}^k \text{ integer}\}.$$

It would be interesting to take $X = Y$, but then the subproblem in y (*i.e.*, $\underset{y \in X_2 \cap Y}{Min} \{u^T y\}$) is as difficult as the original problem to deal with.

The second decomposition (D_2) is obtained from (D_1) by replacing X_1 by a smaller set:

$$X'_1 = \{x : \text{ (2.1) and (2.2) are verified and } x \geq 0\}.$$

Since $X'_1 \subset X_1$, it follows that for all u

$$v_{D_1}(u) \leq v_{D_2}(u) . \tag{4.1}$$

Two other decompositions (D_3) and (D_4) are obtained from (D_1) and (D_2), respectively, by replacing Y by Y'

$$Y' = \{y : 0 \leq y_{ij}^k \leq 1\}.$$

The first Theorem identifies cases where the bound generated with Lagrangean decomposition is no better than the optimal value $v(\overline{P})$ of $(\overline{P})$ generated from (P) by relaxing the integrality constraints on the variables.

THEOREM 4.1

i) $\underset{u}{Max}\ v_{D_3}(u) = v(\overline{P})$.

ii) If $K = 1$, then $\underset{u}{Max}\ v_{D_4}(u) = v(\overline{P})$.

PROOF

i) Since (D_3) is a Lagrangean relaxation of a problem equivalent to (P), it follows from [10, Theorem 1.a] that:

$$\underset{u}{Max}\ v_{D_3}(u) \geq v(\overline{P}).$$

On the other hand, from the definition of X_1 it follows that:

$$\underset{x \in X_1 \cap X}{Min}\ (c - u)^T x = \underset{x \in X_1}{Min}\ (c - u)^T x$$

Indeed, the extreme points of $X_1 \cap X$ coincide with those of X_1. Hence, (D_3) can be seen as a Lagrangean decomposition of $(\overline{P})$, and

$$\underset{u}{Max}\ v_{D_3}(u) \leq v(\overline{P}).$$

ii) The same argument can be used in this case since, under the assumption that $K = 1$, the extreme points of $X'_1 \cap X$ coincide with those of X'_1 (when $K > 1$, it is known that the extreme points of $X'_1 \cap X$ do not coincide with those of X'_1).

□

In [6] , Desrosiers et al. use Lagrangean relaxation to deal with the vehicle scheduling problem with a unique type of vehicle (i.e., K = 1). Now, we introduce two Lagrangean relaxations generalizing to $K \geq 1$ the most efficient relaxation given in [6].

The first Lagrangean relaxation (R_1) is specified relatively to constraints (2.1) and the vector of multipliers λ:

$(PR_{1\lambda})$

$$v_{R_1}(\lambda) = \sum_j \lambda_j + \underset{x \in X_2 \cap Y}{Min} \sum_{k=1}^{K} \sum_{(i,j) \in A_k} (c_{ij}^k - \lambda_j)\, x_{ij}^k \quad .$$

Hence, the constraint set of $(PR_{1\lambda})$ is identical to the subproblem found in the decomposition (D_1) and (D_2). Furthermore, (R_1) is a straightforward extension of a Lagrangean relaxation given in [6] for the case K = 1.

Another relaxation (R_2) is derived from an equivalent formulation (P') of (P):

$$(P') \qquad \text{Min} \quad \sum_{k=1}^{K} \sum_{(i,j) \in A_k} c_{ij}^k\, x_{ij}^k$$

$$\text{Subject to} \quad \sum_{i \in P_j^k} x_{ij}^k = y_j^k, \qquad j \in N_k - \{s,t\},\ 1 \leq k \leq K \qquad (4.2)$$

$$\sum_{k=1}^{K} y_j^k = 1, \qquad j \in N_k - \{s,t\}$$

$$x \in X_2 \cap X,$$

$$y_j^k = 0 \text{ or } 1, \qquad j \in N_k - \{s,t\},\ 1 \leq k \leq K \ .$$

Specify a Lagrangean relaxation (R_2) of (P') relative to the constraints (4.2) and the associated vector of multipliers γ :

$(P'R_{2\gamma})$

$$v_{R_2}(\gamma) = Min \sum_{k=1}^{K} \sum_{j \in N_k - \{s,t\}} \gamma_j^k \, y_j^k + \underset{x \in X_2 \cap Y}{Min} \sum_{k=1}^{K} \sum_{(i,j) \in A_k} (c_{ij}^k - \gamma_j^k) \, x_{ij}^k$$

$$\text{Subject to} \quad \sum_{k=1}^{K} y_j^k = 1, \; j \in N_k - \{s,t\}$$

$$y_j^k = 0 \text{ or } 1, \; j \in N_k - \{s,t\}$$

$$1 \le k \le K .$$

Note that (R_1) and (R_2) are identical whenever K = 1 and reduce to the relaxation introduced in [6]. The following Theorem establishes relations between bounds generated with the decomposition (D_1) and (D_2) and the relaxations (R_1) and (R_2).

THEOREM 4.2

$$\underset{\lambda}{Max} \, v_{R_1}(\lambda) \le \underset{\gamma}{Max} \, v_{R_2}(\gamma) \le \underset{u}{Max} \, v_{D_1}(u) \le \underset{u}{Max} \, v_{D_2}(u).$$

PROOF

To show that $\underset{\lambda}{Max} \, v_{R_1}(\lambda) \le \underset{\gamma}{Max} \, v_{R_2}(\gamma)$, consider the vector of multipliers $\gamma(\lambda)$ for (R_2) where $\gamma(\lambda)_j^k = \lambda_j$, $1 \le k \le K$. Referring to $(P'R_{2\gamma})$, it is easy to see that for all λ,

$$v_{R_2}(\gamma(\lambda)) = v_{R_1}(\lambda).$$

Hence,

$$\underset{\lambda}{Max} \, v_{R_1}(\lambda) = \underset{\lambda}{Max} \, v_{R_2}(\gamma(\lambda)) \le \underset{\gamma}{Max} \, v_{R_2}(\gamma).$$

A similar argument is used to show that $\underset{\gamma}{Max}\ v_{R_2}(\gamma) \leq \underset{u}{Max}\ v_{D_1}(u)$. Indeed, consider the vector of multipliers $(c-u(\gamma))$ for (D_1) where $u(\gamma)_{ij}^k = \gamma_j^k$ for all $i \in N_k$. Referring to $(P'R_{2\gamma})$, it is easy to see that for all γ,

$$v_{R_2}(\gamma) = v_{D_1}(c-u(\gamma)).$$

Hence,

$$\underset{\gamma}{Max}\ v_{R_2}(\gamma) = \underset{\gamma}{Max}\ v_{D_1}(c-u(\gamma)) \leq \underset{u}{Max}\ v_{D_1}(u).$$

□

Theorem 3.1 allows to identify a class of problems for which Lagrangean decompositions do not generate better lower bounds than Lagrangean relaxations. In the next Theorem, we show that our vehicle scheduling problem belongs to this class, and we strengthen the results proved in Theorem 4.2.

THEOREM 4.3

i) If K = 1 (problem with a unique type of vehicle), then
$\underset{\lambda}{Max}\ v_{R_1}(\lambda) = \underset{\gamma}{Max}\ v_{R_2}(\gamma) = \underset{u}{Max}\ v_{D_1}(u) = \underset{u}{Max}\ v_{D_2}(u).$

ii) If K > 1 (problem with more than one type of vehicle), then
$\underset{\lambda}{Max}\ v_{R_1}(\lambda) = \underset{\gamma}{Max}\ v_{R_2}(\gamma) = \underset{u}{Max}\ v_{D_1}(u) \leq \underset{u}{Max}\ v_{D_2}(u).$

PROOF

First we show that, for any value of $K \geq 1$, the constraints set $X_2 \cap Y$ can be assumed to be compact. Indeed, since the number of vehicles of type k required to execute tasks in N_k can be bounded by $|N_k - 2|$, it follows that for all $x \in X_2 \cap Y$, $x_{ij}^k \leq |N_k - 2|$, $(i,j) \in A_k$, $1 \leq k \leq K$. Hence, $X_2 \cap Y$ can be assumed to be compact without any loss of generality.

Furthermore, since the extreme proints of $X_1 \cap X$ coincide with those of X_1, it follows that $CO(X_1 \cap X) = CO(X_1)$.

Now, referring to Theorem 3.1, the compacity of $X_2 \cap Y$ imply that:

$$\underset{u}{Max}\ v_{D_1}(u) = \underset{\lambda}{Max}\ v_{R_1}(\lambda).$$

Hence, referring to (4.1) and Theorem 4.2, it follows that for all $K \geq 1$,

$$\underset{\lambda}{Max}\ v_{R_1}(\lambda) = \underset{\gamma}{Max}\ v_{R_2}(\gamma) = \underset{u}{Max}\ v_{D_1}(u) \leq \underset{u}{Max}\ v_{D_2}(u).$$

If K = 1, then the constraints set $X'_1 \cap X$ (used to specify (D_2)), reduces to:

$$X'_1 \cap X = \{x : \sum_i x_{ij} = 1,\ j \neq s,t\ ;\ \sum_j x_{ij} = 1,\ i \neq s,t\ ;\ x_{ij} = 0,\ 1,\quad i\ ,\ j \in N\}.$$

In this case the extreme points of $X'_1 \cap X$ coincide with those of X'_1, and it follows that $CO(X'_1 \cap X) = CO(X'_1)$. Hence, referring to Theorem 3.1.

$$\underset{\lambda}{Max}\ v_{R_1}(\lambda) = \underset{u}{Max}\ v_{D_2}(u).$$

□

When K is arbitrary, the results indicate that decomposition (D_2) may generate a better bound than relaxations (R_1) and (R_2). On the other hand, the supplementary effort to obtain this improvement (if any) is noticeable. Indeed, to deal with (D_2), two subproblems have to be solved, the first ($\underset{y \in X_2 \cap Y}{Min}\ \{u^T y\}$) being similar to $(PR_{1\lambda})$ and the other ($\underset{x \in X'_1 \cap X}{Min}\ \{(c-u)^T x\}$) being a multicommodity flow problem.

If K = 1, then the bound cannot be improved with Lagrangean decomposition.

5. DUALITY GAP

Whenever Lagrangean decomposition or relaxation are used, the duality gap is an important issue. In this section we analyze how to eliminate duality gap for some problems using the augmented Lagrangean approach.

To derive the main results, consider the following problem:

(M)

$$\begin{aligned} \text{Min} \quad & f(x) \\ \text{Subject to} \quad & F(x) = 0 \\ & x \in X \end{aligned}$$

where $X \subset R_n$ is a polyhedral set and $F(x) = [f_1(x), \ldots, f_m(x)]^T$ is a vector of linear functions f_i. As usual, define the perturbation function associated with (M), (see [10])

$$\begin{aligned} \phi(b) = \quad \text{Min} \quad & f(x) \\ \text{Subject to} \quad & F(x) = b \\ & x \in X . \end{aligned}$$

Hence, $\phi(0) = v(M)$ is the optimal value of (M). A vector α is a subgradient (see [10]) of ϕ at 0 if for all b

$$\phi(b) \geq \phi(0) + \alpha^T b. \tag{5.1}$$

Geoffrion in [10] gives the following conditions for the duality gap to be zero.

THEOREM 5.1 [10, Theorem 3]

Consider the Lagrangean relaxation relative to the constraints $F(x) = 0$ and the associated vector of multipliers λ:

$$(MR_\lambda) \qquad v_R(\lambda) = \underset{x \in X}{Min}\ f(x) + \lambda^T F(x).$$

Then, $\underset{\lambda}{Max}\ v_R(\lambda) = v(P)$ if and only if there exists a global subgradient of ϕ at $b = 0$.

This result is now use to derive conditions in terms of augmented Lagrangean. Consider the following problem:

(M_ρ)

$$\begin{aligned} \text{Min} \quad & f(x) + \rho\, \|F(x)\|^2 \\ \text{Subject to} \quad & F(x) = 0 \\ & x \in X\ . \end{aligned}$$

For any value of $\rho > 0$, $v(M_\rho) = v(M)$. Let $\phi_\rho(b)$ be the perturbation function associated with (M_ρ).

$$\begin{aligned} \phi_\rho(b) = Min \quad & f(x) + \rho\|F(x)\|^2 = \rho\|b\|^2 \quad + Min\ f(x) \\ \text{Subject to} \quad & F(x) = b \\ & x \in X. \end{aligned}$$

Hence,

$$\phi_\rho(b) = \rho\, \|b\|^2 + \phi(b). \tag{5.2}$$

The following result gives conditions for the duality gap to be zero.

THEOREM 5.2

Assume that $\bar{\rho} > 0$ exists where:

$$\bar{\rho} = \underset{\substack{b \neq 0 \\ \phi(b) < \infty}}{Max} \left\{ \frac{\phi(0) - \phi(b)}{\|b\|^2} \right\}. \tag{5.3}$$

Then for all $\rho \geq \bar{\rho}$,

$$v_{R_\rho}(\lambda) = \underset{x \in X}{Min} \{f(x) + \lambda^T F(x) + \rho \|F(x)\|^2\} = v(M_\rho) = v(M).$$

PROOF

First we show that 0 is a global subgradient of ϕ_ρ at $b = 0$ whenever $\rho \geq \bar{\rho}$. Indeed, denote $B = \{b : b \neq 0, \phi(b) < \infty\}$. By definition of $\bar{\rho}$ in (5.3), it follows that for all $b \in B$.

$$\rho \geq \bar{\rho} \geq \frac{\phi(0) - \phi(b)}{\|b\|^2}$$

$$\rho \|b\|^2 \geq \phi(0) - \phi(b)$$

$$\phi(b) + \rho \|b\|^2 \geq \phi(0) + 0^T b.$$

Hence, 0 is a global subgradient of ϕ_ρ at b = 0, and the resut follows from Theorem 5.1.

□

For the vehicle scheduling problem, condition (5.3) is verified for all the Lagrangean decompositions and relaxations introduced in Section 4 since set B is finite in these cases.

Recall that the Lagrangean decompositions of (P) are defined using Lagrangean relaxations relative to the constraints $y - x = 0$ (corresponding to $F(x)$ *in* (M)). In decompositions (D_1) or (D_2),

$$x \in X_1 \cap X \text{ or } x \in X'_1 \cap X \text{ implies that } \quad x_{ij}^k = 0 \text{ or } 1,\ (i,j) \in A_k,\ 1 \le k \le K,$$
$$y \in X_2 \cap Y \text{ implies that } \quad y_{ij}^k \text{ integer},\ (i,j) \in A_k,\ 1 \le k \le K.$$

Furthermore, referring to Theorem 4.3, $X_2 \cap Y$ can be assumed to be compact. Hence, there is a finite number of vectors b such that the associated perturbed problem is feasible. It follows that B is finite.

The argument is even simplified for decompositions (D_3) and (D_4) since:

$$y \in X_2 \cap Y' \text{ implies that } y_{ij}^k = 0 \text{ or } 1,\ (i,j) \in A_k,\ 1 \le k \le K.$$

Similar arguments are used to show that B is a finite set for the Lagrangean relaxations (R_1) and (R_2).

From Theorem 5.2, it follows that the duality gap can be reduced to zero if an augmented version of any Lagrangean decomposition or relaxation is used with a penalty factor ρ large enough.

In [6] Desrosiers et al. use the augmented Lagrangean relaxation (R_1) to deal with the problem where a unique type of vehicle is available ($K = 1$). In all their test problems, they observe that the duality gap was zero. In fact this follows from Theorem 5.2 as illustrated in the preceding paragraph.

REFERENCES

[1] L. Bodin, A. Assad, M. Ball and B. Golden.
The state of the art in routing and scheduling of vehicles and crews.
Computers and operations research 10 (1983) 69-211.

[2] J. Desrosiers, J.A. Ferland, J.M. Rousseau, G. Lapalme and L. Chapleau.
Transcol: A multi period school bus routing and scheduling system.
TIMS, Studies in Management Sciences 22
(E.J. Ignall, A.J. Swersey Eds., North-Holland, Amsterdam) (1986) 47-71.

[3] J. Desrosiers, F. Soumis and M. Desrochers.
Routing with time windows by column generation.
Network 14 (1984) 545-565.

[4] J. Desrosiers, F. Soumis, M. Desrochers and M. Sauve.
Routing and scheduling with time windows solved by network relaxation and branch-and-bound on time variables.
Computer scheduling of Public Transport II
(J.M. Rousseau Ed., North-Holland, Amsterdam) (1985) 451-471.

[5] J. Desrosiers, F. Soumis and M. Sauve.
Lagrangean relaxation for routing with time windows.
Operational Research
(J.P. Brans Ed., North-Holland, Amsterdam) (1984) 673-684.

[6] J. Desrosiers, F. Soumis and M. Sauve.
Lagrangean methods for solving the minimal M-TSP with time windows.
Management science 34 (1988) 1005-1022.

[7] J.A. Ferland. The single depot vehicle scheduling problem.
Recent Advances in System Modeling and Optimization,
Lecture Notes Control Inf. Sci. 87 (1986) 99-112.

[8] J.A. Ferland and P. Michelon. The scheduling vehicle problem with many vehicle types.
Operational Research 39 (1988) 577-584.

[9] M.L. Fisher. The Lagrangian relaxation method for solving integer programming problems.
Management Sciences 27 (1981) 1-18.

[10] A.M. Geoffrion. Lagrangian relaxation for integer programming. *Mathematical Programming Study 2* (1976) 82-116.

[11] D. Graham and L.W. Nuttle. A comparison of heuristics for a school bus scheduling problem. *Transportation Research B 20* (1986) 175-182.

[12] M. Guignard and S. Kim. Lagrangean decomposition: A model yielding stronger Lagrangean bounds. *Mathematical Programming 39* (1987) 215-228.

[13] A. Levin. Scheduling and fleet routing models for transportation systems. *Transportation Sciences 5* (1971) 232-255.

[14] C. Orloff. Route constrained fleet scheduling. *Transportation Science 10* (1976) 149-168.

[15] A.J. Swersey and W. Ballard. Scheduling school buses. *Management Science 30* (1984) 844-853.